McGill University — December 2013
Faculty of Engineering — Final examination

Solution for Final Exam

Math 263

Wednesday, December 18, 2013

Time: 2pm-5pm

Examiner: Prof. J.J. Xu — Associate Examiner: A. Hundmer

Student name (last, first)	**Student number (McGill ID)**

INSTRUCTIONS

1. This is a closed book exam. No notes allowed.
2. Faculty Standard Calculators and Translation Dictionaries are permitted.
3. Before you begin, please take a couple of minutes to scan the problems.(Please inform the invigilator if the booklet is defective).
4. You are expected to show all your work.

This exam comprises the cover page, ten pages of the question numbered 2 to 11, four blank pages numbered 12 to 15, and the Table of Laplace Transforms.

Problem	1	2	3	4	5	6	7	8	9	10 (bonus)	Total
Mark											
Out of	10	10	10	10	10	10	10	15	15	10	100

1. (**10 points**) Consider the equation

$$y' + ay = be^{-\lambda t}$$

where a and λ are positive real constants, and b is a real constant.

(a) Find the homogeneous, particular and general solution when $a \neq \lambda$.

(b) Do the same when $a = \lambda$.

(c) Show that in both cases the general solution has the property that $y \to 0$ as $t \to \infty$.

Solution: Apply the integrating factor $\mu(t) = \mathrm{e}^{at}$. The equation then becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{at}y\right) = b\mathrm{e}^{(a-\lambda)t}$$

which can be integrated to

$$y = \frac{1}{\mathrm{e}^{at}}\left(\int be^{(a-\lambda)t}\mathrm{d}t + C\right).$$

If $a \neq \lambda$, the general solution is therefore

$$y = \frac{b}{a-\lambda}\mathrm{e}^{-\lambda t} + C\mathrm{e}^{-at},$$

clearly consisting of a particular solution plus a homogeneous solution. If $a = \lambda$, the general solution is

$$y = bt\mathrm{e}^{-\lambda t} + C\mathrm{e}^{-at}.$$

Both solutions have negative exponents (since it was given that $a, \lambda > 0$) and hence go to 0 when $t \to \infty$.

2. (**10 points**) Solve the differential equation

$$y + (2xy - e^{-2y})y' = 0.$$

Solution: The equation is not exact but can be made exact by using an integrating factor. One can find μ has to satisfy

$$y\frac{\partial \mu}{\partial y} + \mu = \left(2xy - e^{-2y}\right)\frac{\partial \mu}{\partial x} + 2y\mu.$$

Assuming $\mu = \mu(y)$, one can find easily that

$$\mu = \frac{1}{y}e^{2y}.$$

After multiplying with this factor,

$$M = e^{2y}, \quad N = 2xe^{2y} - \frac{1}{y}.$$

Hence

$$F(x,y) = \int M dx = xe^{2y} + h(y).$$

From $\partial F/\partial y = N$, one obtains that $h'(y) = -1/y$ and therefore $h(y) = -\ln|y|$. The solution is consequently

$$xe^{2y} - \ln|y| = C.$$

3. (**10 points**) Consider the equation

$$yy'' + (y')^2 = 0, \quad (t > 0).$$

(a) Accurately classify this ordinary differential equation.

(b) Show that $y_1(t) = 1$ and $y_2(t) = \sqrt{t}$ are linearly independent solutions.

(c) For arbitrary constants c_1 and c_2, does $c_1 y_1 + c_2 y_2$ solve the given equation? Why or why not?

(d) Find the general solution of above equation by writing the left hand side as the derivative of a product.

Solution: This is a nonlinear second order differential equation. By plugging in, one can check easily that $y_1(t) = 1$ and $y_2(t) = \sqrt{t}$ are solutions. The Wronskian is

$$W\{y_1, y_2\}(t) = \frac{1}{2\sqrt{t}}$$

and thus the solutions are linearly independent. Linear combinations, however, are not necessarily solutions since the equation is not linear and the superposition principle does not hold. The solution can be found by rewriting the differential equation into

$$\frac{\mathrm{d}}{\mathrm{d}t}(yy') = 0.$$

After integration one obtains the first order equation

$$yy' = C$$

which is separable and leads to

$$\frac{y^2}{2} = Ct + D.$$

We conclude that the general solution is

$$y = \pm\sqrt{Ct + D}.$$

4.(**10 points**) Consider solving the equation

$$(D^2 - I)(D - I)y = e^{-t} + \cos(t)$$

by using the method of differential operators.

(a) Determine the annihilator of the inhomogeneous term and the form of the particular solution.

(b) Find the particular solution and the general solution for the equation.

Solution: The annihilator of the inhomogeneous term is

$$Q(D) = (D + I)(D^2 + I).$$

The differential operator $P(D) = (D + I)(D - I)^2$ and $Q(D)$ have the factor $(D + I)$ in common with the index $m = 1$. So that, one has the particular solution

$$y_p(x) \in \Big\{ t\ker\{(D + I)\} + \ker\{(D^2 + I)\} \Big\}.$$

hence, if has the form

$$y_p = Ate^{-t} + B\cos(t) + C\sin(t).$$

Plugging this into the equation, one obtains

$$y_p = \frac{1}{4}te^{-t} + \frac{1}{4}\cos(t) - \frac{1}{4}\sin(t).$$

The general solution is consequently

$$y = c_1e^t + c_2te^t + c_3e^{-t} + \frac{1}{4}te^{-t} + \frac{1}{4}\cos(t) - \frac{1}{4}\sin(t).$$

5. (**10 points**) By making a change of variables, and by using the basic equalities for the differential operator method **carefully**, find the general solution of the equation

$$x^2y'' + xy' - y = \ln(x), \quad (x > 0).$$

Solution: We change the independent variable x to t, by letting $x = \mathrm{e}^t$. Then the equation reduces to an equation for $\tilde{y}(t) = y(x)$, namely

$$\frac{\mathrm{d}^2\tilde{y}}{\mathrm{d}t^2} - \tilde{y} = t.$$

This equation can be reduced to

$$(D - I)(D + I)\tilde{y} = t.$$

Defining $z = (D + I)\tilde{y}$, we have $(D - I)z = t$. Using the fundamental identity, we rewrite this into

$$\mathrm{e}^tD(\mathrm{e}^{-t}z) = t.$$

The solution is found by integration to be

$$z = -t - 1 + c_1\mathrm{e}^t.$$

Then

$$\mathrm{e}^{-t}D(\mathrm{e}^t\tilde{y}) = -t - 1 + c_1\mathrm{e}^t.$$

Integration returns

$$\tilde{y} = -t + c_1\mathrm{e}^t + c_2\mathrm{e}^{-t}.$$

Finally, the answer for the original problem is

$$y = -\ln x + c_1x + \frac{c_2}{x}.$$

6. (***10 points**)
(1) Compute the Laplace transform of the piece-wise continuous function $f(t)$:

$$f(t) = \begin{cases} t, & 0 \le t < 1; \\ \sin t, & 1 \le t < 2; \\ \mathrm{e}^t, & 2 \le t < 3; \\ 0, & t \ge 3. \end{cases}$$

(2). Find the inverse Laplace transform of the function:

$$F(s) = \frac{2s - 3}{s^2 + 2s + 10}.$$

Solution:
(1)
$$f(t) = t - t\,u_1(t) + [u_1(t) - u_2(t)]\sin t + [u_2(t) - u_3(t)]\mathrm{e}^t;$$
$$\mathcal{L}\{f(t)\} = \frac{1}{s^2} - \frac{\mathrm{e}^{-s}}{s^2} - \frac{\mathrm{e}^{-s}}{s} + \frac{\cos 1 + s\sin 1}{s^2+1}\mathrm{e}^{-s} - \frac{\cos 2 + s\sin 2}{s^2+1}\mathrm{e}^{-2s} + \frac{1}{(s-1)}(\mathrm{e}^{2-2s} - \mathrm{e}^{3-3s})$$
(2)
$$\mathcal{L}^{-1}\{F(s)\} = 2\mathrm{e}^{-t}\cos 3t - \frac{5}{3}\mathrm{e}^{-t}\sin 3t.$$

Solution:
7. (***10 points**) Solve the following initial value problem:
$$y'' + 3y' + 2y = \delta(t-5) + u_{10}(t) \quad y(0) = 0\,, \quad y'(0) = 1/2.$$

Solution:
$$y(t) = -\frac{1}{2}\mathrm{e}^{-2t} + \frac{1}{2}\mathrm{e}^{-t} + u_5(t)\left[-\mathrm{e}^{-2(t-5)} + \mathrm{e}^{-(t-5)}\right] + u_{10}(t)\left[\frac{1}{2} + \frac{1}{2}\mathrm{e}^{-2(t-10)} - \mathrm{e}^{-(t-10)}\right].$$

8.(***15 points**) Given Eq.:
$$xy'' + y' - y = 0, \quad (x > 0).$$

1. Show that $x = 0$ is a regular singular points, derive the corresponding *indicial equation* and give the form of the series solutions near $x = 0$;
2. Derive the *recurrence formula* of the coefficients and determine the forms of two linear independent solutions: $\{y_1(x), y_2(x)\}$.
3. Write the complete expressions of $\{y_1(x), y_2(x)\}$, and determine the radii $\{\rho_1, \rho_2\}$ of convergence of these two series solutions.

solution:
$$F(r) = r^2, \quad a_n = \frac{a_{n-1}}{(n+r)^2}.$$
As $n \geq 1$):
$$\hat{a}_n(r) = \frac{1}{(1+r)^2\cdots(n+r)^2}; \quad \hat{a}_n'(r) = -2\hat{a}_n(r)\left[\frac{1}{(1+r)} + \cdots + \frac{1}{(n+r)}\right]$$
Hence,
$$\hat{a}_n(0) = \frac{1}{(n!)^2}; \quad \hat{a}_n'(0) = -\frac{2H_n}{(n!)^2}$$
where
$$H_n = \left[1 + \frac{1}{2} + \cdots + \frac{1}{n}\right]$$
Therefore, we have
$$y_1(x) = 1 + \sum_{n=1}^{\infty}\frac{x^n}{(n!)^2} = \mathrm{e}^x, \quad \rho_1 = \infty;$$
$$y_2(x) = -2\sum_{n=1}^{\infty}\frac{H_n x^n}{(n!)^2}, \quad \rho_2 = \infty.$$

9.(***15 points**)

(1). Find all values of β for which all solutions of Eq.

$$x^2y'' + \beta y = 0$$

approach zero as $x \to 0$.
(2). Find γ so that the solution of the initial value problem:

$$x^2y'' - 2y = 0; \quad y(1) = 1, y'(1) = \gamma$$

is bounded as $x \to 0$.

Solution: (1). This is Euler equation: Solutions:

$$y(x) = c_1x^{r_1} + c_2x^{r_2}$$

where

$$r(r-1) + \beta = 0, \quad r_{1,2} = \frac{-1 \pm \sqrt{1-4\beta}}{2}$$

To ensure $y(x) \to 0$ as $x \to 0$, it is required that $\Re\{r_{1,2}\} > 0$. It follows that (i). For complex roots: $1 - 4\beta < 0$, or $\beta > \frac{1}{4}$. (ii). For real roots: $1 - \sqrt{1-4\beta} > 0$, or $1 - 4\beta > 0$ and $\beta > 0$. Thus, we derive the condition: $\beta > 0$.
(2) With

$$r(r-1) - 2 = 0, \quad r_1 = -1, r_2 = 2,$$

we derive the general solution:

$$y(x) = c_1x^{-1} + c_2x^2.$$

To ensure $y(x) \to 0$ as $x \to 0$, we derive $c_1 = 0$. Furthermore, from the IC', we derive $c_2 = 1$, and $y'(1) = 2 = \gamma$.

10.(***10 points**) Given the system of linear equations:

$$\mathbf{x}'(t) = A\mathbf{x}, \quad \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = (A) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}.$$

1. Find all the eigenvalues of the matrix A and determine the eigenvector corresponding each of the eigenvalues.

2. Find the general solutions of the system of equations.

3. Determine the behavior of solutions.

Solution:
(1). The characteristic polynomial: $\phi(\lambda) = \det(A - \lambda I) = \lambda^2 - 3\lambda - 4$; $\lambda_1 = -1, \lambda_2 = 4$. For $\lambda_1 = -1$, $\hat{\mathbf{v}}_\mathbf{1} = (-1, 2)^T$; For $\lambda_2 = 4$, $\hat{\mathbf{v}}_\mathbf{2} = (2, 1)^T$.
(2). General solution:

$$\mathbf{x}(t) = c_1\hat{\mathbf{v}}_\mathbf{1}\mathrm{e}^{-t} + c_2\hat{\mathbf{v}}_\mathbf{2}\mathrm{e}^{4t}.$$

(4). Laplace transform method.

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